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Reflexive Modules

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INTRODUCTION AND PRELIMINARIES

The first systematical treatment of reflexive modules over commutative rings was done by Bourbaki in [7, 4]. He considered reflexive modules of finite type over a normal domain A, and generalized classical results about modules of finite type over Dedikind domains. Some special classes of reflexive modules had been previously considered. In particular, it is well known that the set of all reflexive modules of rank 1 over the domain A has a natural group structure. This group (called the Divisor Class group) is an important invariant of the ring and plays an essential role in algebraic geometry.

Little is known about reflexive modules of rank greater than one, although it seems certain that they should play an important role in studying both global and local properties of algebraic varieties. In the present paper we generalize some known facts about reflexive modules and demonstrate the relations between them, the Brauer group, and vector bundles. It would be nice to have some geometric structure on some subsets of the set of reflexive modules. We shall return to this problem in a future paper.

The paper consists of four sections. In the first section we generalize results of Bourbaki on reflexive modules. In particular, we show how far his results can be extended. In the second section we study reflexive modules over local normal domains for which the ring of endomorphisms is a free module. In the third section we study reflexive modules over two-dimensional complete local normal domains. The last section is a complement to the second. There we compare the Brauer group of a two-dimensional singularity with the Brauer group of the surface obtained by a desingularization.

Each section is preceded by a short introduction to the results of that section.

We now recall the definition of a reflexive module.

DEFINITION. Let A be a Krull domain and M an A-lattice in a finite-dimensional vector space over the quotient field of A [15, I]. The module M is called reflexive if one of the following two equivalent conditions is satisfied:

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(i) $M \simeq M^{**}$, where $M^* = \mathrm{Hom}_A(M, A)$,

(ii) $M = \bigcap_{ht\mathfrak{p}} M_{\mathfrak{p}}$, where $\mathfrak{p} \subset A$ are prime ideals.

Now let R be any commutative ring and M an R-module. Suppose that M^* is an R-module of finite type. Then we have an exact sequence

$$F_2 \to F_1 \to M^* \to 0,$$

where F_1 , F_2 are free R-modules and F_1 is of finite type. This sequence yields an exact sequence

$$0 \to M^{**} \to F_1^* \to F_2^*.$$

Therefore, if in addition $M \approx M^{**}$, then M is the kernel of a map of two free R-modules of finite type.

On the other hand, if M is a lattice over a Krull domain A and if there exists an exact sequence of A-modules

$$0 \to M \to L_1 \to L_2$$

with L_1 , L_2 free modules of finite type, then M is a reflexive A-module.

Throughout the paper a normal domain will mean a Noetherian integrally closed domain. By the Class group of a normal domain we mean its Divisor Class group. If T is a module over a commutative ring B, then we denote by $T^{\sim}$ the corresponding quasi-coherent sheaf on Spec(B). All schemes considered in this paper are separated.

1. Coherent Sheaves on Subspaces of Codimension $\leqslant 1$ in Schemes

The main results of this section generalize the theorems of Bourbaki on reflexive modules over normal domains. Our results are stated and proved in the sheaf-theoretical language, and some of them are valid for not necessarily affine schemes. A much more general situation was discussed in [30], but, obviously, the theorem proved here cannot be generalized to that case.

(1.1) Let us consider an integral scheme X of dimension $k \geqslant 1$. We shall denote by X_1 the subspace in X of all points of codimension 1. This subspace has a natural structure of a ringed space with a structure sheaf $\mathcal{O}_{X_1} = \mathcal{O}_X \mid X_1$ (the restriction to X_1 of the sheaf $\mathcal{O}_X$). Thus, we may consider sheaves of $\mathcal{O}_{X_1}$-modules.

Let $\mathscr{F}(X)$ be the category of quasi-coherent sheaves on the scheme X. A sheaf N of $\mathcal{O}_{X_1}$-modules is called quasi-coherent if $N = j^*(M)$, where M is a quasi-coherent sheaf on X and $j\colon X_1 \hookrightarrow X$ is the natural embedding. Let $\mathscr{F}(X)^{X_1}$ denote the full subcategory in $\mathscr{F}(X)$ consisting of all $\mathcal{O}_X$-modules M with $j^*(M) = 0$.

DEFINITION (1.2) (cf. [30]). Let X be an integral scheme of dimension $k \geqslant 1$. The scheme X is called 1-Noetherian if there exists a finite open affine covering $\{U_i\}_{i\in I}$ of X such that for all $i \in I$ an object $\mathcal{O}_{U_i}$ is Noetherian in the quotient category $\mathscr{F}(U_i)/\mathscr{F}(U_i)^{(U_i)_1}$.

DEFINITION (1.3). A 1-Noetherian scheme X is called 1-regular if the local rings $\mathcal{O}_x$ are regular for all points $x \in X_1$.

(1.4) Let X be a 1-Noetherian scheme. The category $\mathscr{F}(X_1)$ of all quasi-coherent sheaves on X_1 is Abelian and equivalent to the category $\mathscr{F}(X)/\mathscr{F}(X)^{X_1}$ [30]. The Noetherian objects of $\mathscr{F}(X_1)$ are called coherent sheaves on X_1. In [18, No. 4, Chap. 0, 5.1.3, 5.3.1] we have another definition of coherent (and quasi-coherent) sheaves on a ringed space. It follows immediately from [30] that in our case the two definitions are equivalent.

We shall denote by $\mathscr{F}(X_1)^{\#}$ (resp. $\mathscr{F}^{\#}(X)^{X_1}$) the full subcategory in $\mathscr{F}(X_1)$ (resp. $\mathscr{F}(X)^{X_1}$) of all coherent sheaves.

The scheme X will often satisfy the following condition:

(1.5) There exists an invertible $\mathcal{O}_{X_1}$-module L such that for any coherent $\mathcal{O}_{X_1}$-module M there is a positive integer $n_0(M)$ such that $M \otimes_{\mathcal{O}_{X_1}} L^{\otimes n}$ is generated by global sections for $n \geqslant n_0(M)$ [18, No. 8, 11, (4.5)].

THEOREM (1.6). *Let $X = \operatorname{Spec}(A)$ be a 1-Noetherian affine scheme. The following conditions are equivalent:*

(i) *For any torsion free sheaf $M \in \mathscr{F}(X_1)^{\#}$ there exists an exact sequence of $\mathcal{O}_{X_1}$-modules*

$$0 \to \mathcal{O}_{X_1}^n \to M \to N \to 0,$$

where $N \in \mathscr{F}(X_1)^{\#}$ is a torsion free sheaf of rank 1.

(ii) *Any object $E \in \mathscr{F}(X_1)^{\#}$ is a direct sum $E = M \oplus T$, where M is a torsion free subsheaf and T is a torsion subsheaf, M, $T \in \mathscr{F}(X_1)^{\#}$.*

(iii) *X is a 1-regular scheme.*

The proof is contained in (1.7)–(1.11). Actually, most of the theorem (except (ii), (iii) implies (i)) is proved for any (not necessarily affine) 1-Noetherian scheme (see Remark (1.12) below).

PROPOSITION (1.7). *Let X be any (not necessarily affine) 1-regular scheme. Then any object of the category $\mathscr{F}(X_1)^{\#}$ is a direct sum of a torsion sheaf and a torsion free sheaf from $\mathscr{F}(X_1)^{\#}$. Furthermore, any torsion sheaf of the category $\mathscr{F}(X_1)^{\#}$ can be written in a unique way as the finite direct sum*

$$\bigoplus_i g_*^i(\mathcal{O}_i/\mathfrak{m}_i^{s_i}),$$

where the $\mathcal{O}_i$ *are local rings of the form* $\mathcal{O}_x$ (codim$(x) = 1$) *with maximal ideals* $m_i \in \mathcal{O}_i$, g^i: Spec$(\mathcal{O}_i) \hookrightarrow X_1$ *the natural embeddings, and* s_i *positive numbers.*

Proof. Let M be an object of the category $\mathscr{F}(X_1)^{\#}$. Let $B \subset M$ be the maximal torsion subobject of M (so that $D = M/B$ is a torsion free sheaf). It is enough to prove that $\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(D, B) = 0$.

As in [16, IV, (4.2)] we can apply the spectral sequence of composition functors

$$H^p(X_1, \mathscr{E}xt^q_{\mathcal{O}_{X_1}}(D, B)) \Rightarrow \mathrm{Ext}^m_{\mathcal{O}_{Y_1}}(D, B),$$

and get the three term exact sequence

$$0 \to H^1(X_1, \mathscr{H}om_{\mathcal{O}_{X_1}}(D, B)) \to \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(D, B) \to H^0(X_1, \mathscr{E}xt^1_{\mathcal{O}_{X_1}}(D, B)).$$

Because the sheaf $\mathscr{H}om_{\mathcal{O}_{X_1}}(D, B)$ has support in a finite number of points, the first term is trivial. From [16, IV, Theorem (4.2.2)] it follows that

$$(\mathscr{E}xt^1_{\mathcal{O}_{X_1}}(D, B))_x \approx \mathrm{Ext}^1_{\mathcal{O}_x}(D_x, B_x) \qquad \text{for any point } x \in X_1.$$

The scheme X is 1-regular and the D_x, $x \in X_1$, are torsion free, hence free $\mathcal{O}_x$-modules, so $\mathrm{Ext}^1_{\mathcal{O}_X}(D_x, B_x) = 0$ for all $x \in X_1$. This implies $H^0(X, \mathscr{E}xt^1_{\mathcal{O}_{X_1}}(D, B)) = 0$. Therefore $\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(D, B) = 0$.

Since any torsion coherent sheaf on X_1 has support in a finite number of points, the last assertion of the proposition follows from the well-known characterization of modules of finite type over discrete valuation rings [7, 4, Proposition 23].

Proposition (1.8). *Let* X *be any (not necessarily affine)* 1*-Noetherian scheme satisfying condition* (1.5). *If* X *satisfies one of the following two conditions:*

(a) *for any torsion free sheaf* $M \in \mathscr{F}(X_1)^{\#}$ *of rank* 2 *generated by global sections there exists an exact sequence of* $\mathcal{O}_{X_1}$*-modules*

$$0 \to \mathcal{O}_{X_1} \to M \to N \to 0,$$

where $N \in \mathscr{F}(X_1)^{\#}$ *is a torsion free sheaf of rank* 1;

(b) *any object* $E \in \mathscr{F}(X_1)^{\#}$ *is a direct sum:* $E = M \oplus T$, *where* M *is a torsion free subsheaf and* T *a torsion subsheaf* ($M, T \in \mathscr{F}(X_1)^{\#}$), *then it is a* 1*-regular scheme.*

Proof.

Case (a). Suppose there is $y \in X_1$ such that the ring $\mathcal{O}_y$ is not regular. By the Akidzuki–Cohen theorem [26, Proposition 6], there is an ideal $J \subset \mathcal{O}_y$ with the greatest number of generators among all ideals of $\mathcal{O}_y$. Since the scheme X

satisfies condition (1.5), we can find a coherent torsion free sheaf Q_{X_1} of rank 1 generated by global sections such that $Q_y \approx J$.

We have an exact sequence of coherent $\mathcal{O}_{X_1}$-modules

$$0 \to \mathcal{O}_{X_1} \to Q_{X_1} \oplus Q_{X_1} \to P_{X_1} \to 0,$$

where P_{X_1} is a torsion free $\mathcal{O}_{X_1}$-module of rank 1. This sequence yields an exact sequence of $\mathcal{O}_y$-modules

$$0 \to \mathcal{O}_y \to J \oplus J \to P_y \to 0.$$

Because the number of generators of $J \oplus J$ is greater than $1+$ (the number of generators of P_y), we have a contradiction.

Case (b). To prove that X is a 1-regular scheme, it is enough to show that $\mathrm{Ext}^1_{\mathcal{O}_x}(J, \mathcal{O}_x/\mathfrak{m}_x) = 0$ for any point $x \in X$ of codimension 1 and any ideal $J \subset \mathcal{O}_x$.

Let $x \in X$ be a point of codimension 1. As before, we can find a torsion free $\mathcal{O}_{X_1}$-module $Q_{X_1} \in \mathscr{F}(X_1)^{\#}$ such that $Q_x \approx J$. Our assumptions imply $\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(Q_{X_1}, j_*(\mathcal{O}_x/\mathfrak{m}_x)) = 0$, where $j\colon \mathrm{Spect}(\mathcal{O}_x) \hookrightarrow X_1$. Hence $\mathrm{Ext}^1_{\mathcal{O}_x}(Q_x, \mathcal{O}_x/\mathfrak{m}_x) = 0$.

Lemma (1.9). *Let $X = \mathrm{Spec}(A)$ be a 1-regular affine scheme. Then the ring $S = \bigcap_{ht\mathfrak{p}=1} A_\mathfrak{p}$ is a Krull domain. Let $Y = \mathrm{Spec}(S)$. Then we have a natural isomorphism of ringed spaces: $(Y_1, \mathcal{O}_{Y_1}) \approx (X_1, \mathcal{O}_{X_1})$.*

Proof. Denote by K the quotient field of A. Every discrete valuation ring of the form $A_\mathfrak{p}(\mathfrak{p} \subset A)$ defines a discrete valuation $v_\mathfrak{p}$ of rank 1 of the field K. Since X is 1-Noetherian, for any element $f \in S$, $f \neq 0$, there is at most a finite number of the valuations $v_\mathfrak{p}$ such that f is not a unit in $A_\mathfrak{p}$. Hence S is a Krull domain [15, I, 1]. A trivial verification, which we omit, shows that $(Y_1, \mathcal{O}_{Y_1}) \approx (X_1, \mathcal{O}_{X_1})$ (see also [15, I, 3]).

Lemma (1.10). *Let $X = \mathrm{Spec}(A)$ be a 1-regular affine scheme and $M \in \mathscr{F}(X_1)^{\#}$ a torsion free sheaf of rank 2. Then there exists an exact sequence of $\mathcal{O}_{X_1}$-modules*

$$0 \to \mathcal{O}_{X_1} \to M \to N \to 0,$$

where $N \in \mathscr{F}(X_1)^{\#}$ is torsion-free of rank 1.

Proof. It is enough to prove the assertion for the ring $S = \bigcap_{ht\mathfrak{p}=1} A_\mathfrak{p}$. In this case the proof is the same as in [7, 4, Proof of Lemma 8].

(1.11) Lemma (1.10) and a simple induction (note that X is affine) prove that (iii) implies (i).

This completes the proof of Theorem (1.6).

Remark (1.12). Let X be any 1-Noetherian scheme satisfying condition (1.5). Probably, condition (a) of Proposition (1.8) and conditions (ii), (iii) of

Theorem (1.6) are equivalent. To prove it one must generalize Lemma (1.10) to the nonaffine case. Using Bertini's theorem Maruyama proved an analog of Lemma (1.10) for nonsingular projective surface [25, Lemma 3.1].

Remark (1.13) (cf. [25]). Suppose X is a 1-regular projective variety over algebraically closed field k. One can attempt to construct the moduli scheme of locally free sheaves of finite rank (i.e., vector bundles) on X_1. The main point is to prove some kind of "boundedness" theorem (see [25]). For curves this means that the set of isomorphism classes of indecomposable vector bundles with fixed degree and rank is bounded. If $\dim(X) \geqslant 2$ we must overcome many difficulties. For example, the first cohomology group $H^1(X_1, \mathcal{O}_{X_1})$ fails to be a finite-dimensional vector space over k even for affine X. Maybe, one should consider instead vector bundles on X_2 (points of codimension $\leqslant 2$) and try to generalize recent results of Maruyama [25]. In this case at least we have no problems with the first cohomology groups.

(1.14) Let X be a 1-Noetherian scheme. We recall in brief the definition of the Class group of 1-cycles $\mathrm{Cl}(X)$ [18, No. 32, IV, (26.1)].

Let $\mathfrak{Z}^1(X)$ (resp. $\mathfrak{Z}(X_1)$) denote a free Abelian group generated by the points of codimension 1 (resp. codimension $\leqslant 1$). Let $\mathfrak{Z}^1(X)_{\mathrm{princ}} \subset \mathfrak{Z}^1(X)$ be the subgroup of principal 1-cycles, i.e., sums of the form $\sum O_x(f)\{x\}$ where f is a rational function on X and $O_x(f)$ the order of f at the point $x \in X_1$ of codimension 1.

Definition (1.15). $\mathrm{Cl}(X) = \mathfrak{Z}^1(X)/\mathfrak{Z}^1(X)_{\mathrm{princ}}$.

(1.16) As usual, we denote by $\mathrm{Pic}(Y)$ the Picard group of a ringed space Y. Let D be a positive divisor on X_1 and $Y(D) \subset X_1$ a closed subspace in X_1 defined by the sheaf of ideals $J_{X_1}(D) \subset \mathcal{O}_{X_1}$.

Let $\mathcal{O}_{Y(D)} = \mathcal{O}_{X_1}/J_{X_1}(D) \mid Y(D)$ denote the structure sheaf of the subspace $Y(D)$. The map $cyc\colon D \to \sum \mathrm{length}(\mathcal{O}_{Y(D),x})\{x\}$ induces the map

$$cyc\colon \mathrm{Pic}(X_1) \to \mathrm{Cl}(X_1).$$

Proposition (1.17). *For any 1-regular scheme X the canonical map cyc is an isomorphism of groups.*

The proof of this proposition is essentially due to Grothendieck [18, No. 32, IV, (21.6.12)]. We omit the details.

(1.18) We shall denote by $G(X_1)$ the Grothendieck group of the category $\mathcal{F}(X_1)^{\#}$. Let $g^x\colon \mathrm{Spec}(\mathcal{O}_x) \hookrightarrow X_1$, and let x_0 denote the generic point of X. We have the canonical homomorphism of groups

$$\varphi\colon \mathfrak{Z}(X_1) \to G(X_1),$$

where $\varphi(\{x\}) = [g^x_*(\mathcal{O}_x/\mathfrak{m}_x)]$ if $\mathrm{codim}(x) = 1$, and $\varphi(\{x_0\}) = [\mathcal{O}_{X_1}]$.

PROPOSITION (1.19). *Let $X = \operatorname{Spec}(A)$ be a 1-Noetherian affine scheme. Then the homomorphism φ induces the isomorphism*

$$\mathbf{Z} \oplus \operatorname{Cl}(X) \simeq G(X_1).$$

Proof. We follow closely the idea of Eagon [12]. A prime-filtration Φ of the sheaf $E \in \mathscr{F}(X_1)^{\#}$ is one whose factors are of the form $(A^{\sim}/\mathfrak{p} \mid X_1)$ for various prime ideals $\mathfrak{p} \subset A$ of height $\leqslant 1$. Every sheaf $E \in \mathscr{F}(X_1)^{\#}$ has a prime-filtration. We denote by $n_x(\Phi)$ the number of times that a factor isomorphic to $(A^{\sim}/\mathfrak{p}_x \mid X_1)$, where $\mathfrak{p}_x$ is the prime ideal in the ring A corresponding to the point $x \in X_1$, is associated with Φ. Consider sums of the following form

$$\sum_{\operatorname{codim}(x)=1} (n_x(\Phi) - n_x(\Phi'))\,\{x\},$$

where Φ and Φ' are two arbitrary prime-filtrations of a given sheaf E. Denote by S the subgroup in $\mathfrak{Z}(X_1)$ generated by all such sums.

LEMMA (1.19.1). *The homomorphism φ induces the isomorphism*

$$\mathfrak{Z}(X_1)/S \simeq G(X_1).$$

The proof is the same as in Eagon's paper [12, Lemma].

Exactly as in [12, Corollary 2] we can now reduce the set of generators of S:

LEMMA (1.19.2). *For each $s \in \Gamma(X_1, \mathcal{O}_{X_1})$, $s \neq 0$, choose a particular prime-filtration Φ of $\mathcal{O}_{X_1}/s\mathcal{O}_{X_1}$. Then S is generated by sums $\sum_{\operatorname{codim}(x)=1} n_x(\Phi)\,\{x\}$ quantifying only over $s \in \Gamma(X_1, \mathcal{O}_{X_1})$, $s \neq 0$.*

It follows immediately from Lemma (1.19.2) that the map $\varphi'\colon \mathbf{Z} \oplus \mathfrak{Z}^1(X_1) \to \mathfrak{Z}(X_1)/S$ defined by $\varphi'(l) = l\{x_0\} \pmod S$ for $l \in \mathbf{Z}$, $\varphi'(\{x\}) = \{x\} \pmod S$ for $\{x\} \in \mathfrak{Z}^1(X_1)$ induces the isomorphism

$$\mathbf{Z} \oplus \operatorname{Cl}(X) \simeq \mathfrak{Z}(X_1)/S.$$

Composition of this isomorphism and the isomorphism given in Lemma (1.19.1) gives the isomorphism

$$\mathbf{Z} \oplus \operatorname{Cl}(X) \simeq G(X_1).$$

Remark (1.20). Let $X = \operatorname{Spec}(A)$ be a 1-regular affine scheme. We denote by $K_0(X_1)$ the Grothendieck group of the category of locally free $\mathcal{O}_{X_1}$-modules of finite rank. If M is such a sheaf one can define the sheaf $\det(M) = \Lambda^{rkM}M$ of rank 1. As usual, we get the epimorphism of groups

$$det\colon K_0(X_1) \to \operatorname{Pic}(X_1).$$

Because X is 1-regular, the map

$$i: \operatorname{Pic}(X_1) \to K_0(X_1), \qquad i([P]) = [P] - [\mathcal{O}_{X_1}]$$

is a monomorphism of groups. Indeed, $i([P \otimes Q]) = [P \otimes Q] - [\mathcal{O}_{X_1}]$ and $i([P]) + i([Q]) = [P] + [Q] - 2[\mathcal{O}_{X_1}]$. By Theorem (1.6), there exists an exact sequence of $\mathcal{O}_{X_1}$-modules

$$0 \to \mathcal{O}_{X_1} \to P \oplus Q \to N \to 0,$$

where N is a torsion free $\mathcal{O}_{X_1}$-module. Applying the functor *det* we get an isomorphism $N \approx P \otimes Q$.

It is obvious that $det \cdot i = id$. Define $\epsilon: \mathbf{Z} \to K_0(X_1)$ by $\epsilon(1) = [\mathcal{O}_{X_1}]$. Using Theorem (1.6) one easily verifies that

$$0 \to \mathbf{Z} \xrightarrow{\epsilon} K_0(X_1) \xrightarrow{det} \operatorname{Pic}(X_1) \to 0$$

is a split exact sequence. Hence $K_0(X_1) \approx \operatorname{Pic}(X_1) \oplus \mathbf{Z}$.

Remark (1.21). If $X = \operatorname{Spec}(A)$ is a 1-regular affine scheme, then $\operatorname{Pic}(X_1) \approx \operatorname{Cl}(X)$ and $K_0(X_1) \approx G(X_1)$. Therefore, the previous remark gives another proof of the isomorphism $G(X_1) \approx \operatorname{Cl}(X) \oplus \mathbf{Z}$ in this special case.

PROPOSITION (1.22). *Let A be a (not necessarily Noetherian) local Krull domain. The following conditions are equivalent:*

(i) *Every ideal $J \subset A$ with two generators has finite homological dimension.*

(ii) *The ring A is factorial.*

Proof. To prove (i) implies (ii) it is sufficient to show that $\operatorname{Cl}(X) \approx \operatorname{Pic}(X_1) = 0$, i.e., for any ideal $J \subset A$, $[\det(J^{\sim} \mid X_1)] = 0$ in $\operatorname{Pic}(X_1)$. Let J be an ideal in A. We can find an ideal $Q \subset A$ generated by two elements such that $Q^{\sim} \mid X_1 = J^{\sim} \mid X_1$ [7, 1, Exercise 11]. By assumption, we have a finite resolution

$$0 \to \mathcal{O}_{X_1}^{n_k} \to \cdots \to \mathcal{O}_{X_1}^{n_0} \to (J^{\sim} \mid X_1) \to 0.$$

Hence $[\det(J^{\sim} \mid X_1)] = 0$.

The converse is trivial.

This proposition is a slight generalization of the theorem of Auslander and Buchsbaum [15, II, Theorem 9.1].

(1.123) Theorem (1.6) and Proposition (1.19) imply the results of Bourbaki on reflexive modules over normal domains [7, 4, Proposition 17, Theorems 4 and 6].

2. Reflexive Modules for which the Ring of Endomorphisms is Projective

Throughout this section R will be a local normal domain with maximal ideal $\mathfrak{m}$ and quotient field K. Our main result gives a description of the set of all reflexive R-modules M of finite type for which $\mathrm{End}_R(M)$ is a projective R-module. We conclude the section with two propositions about reflexive modules.

(2.1) Let $\{(S_i, \mathfrak{m}_i)\}_{i \in I}$ be a directed family of Galois coverings with fixed maximal ideals $\mathfrak{m}_i \subset S_i$. By a Galois covering we mean a finite Galois extension. Let us fix one such family. The derived family of local rings $\{(S_i)\mathfrak{m}_i\}_{i \in I}$ is directed. Let $\tilde{R} = \text{inj.}\lim(S_i)\mathfrak{m}_i$. It is easy to see that all the S_i and $\tilde{R}$ are normal domains [11, Proposition 2].

Let π_1 be the fundamental group of the category Et_R of all finite etale coverings of R [28, IV]. We have $\pi_1 = \text{proj}\lim(\pi_1{}^i)$ where $\pi_1{}^i = \mathrm{Aut}_R(S_i)$.

The fundamental group π_1 acts in an obvious way on $\mathrm{Cl}(S_i)$ $(i \in I)$ and $\mathrm{Cl}(\tilde{R})$. Let $\mathrm{Cl}(\tilde{R})^{\pi_1}$ (resp. $\mathrm{Cl}(S_i)^{\pi_1}$) denote the subgroup of fixed element in $\mathrm{Cl}(\tilde{R})$ (resp. $\mathrm{Cl}(S_i)$).

(2.2) We denote by $\mathrm{Br}(B)$ the Brauer group of the ring B and by $\mathrm{Br}(K/R)$ the $\mathrm{Ker}(\mathrm{Br}(R) \to \mathrm{Br}(K))$ [20].

Our main observation (2.6) is that any $[M] \in \mathrm{Cl}(S_i)^{\pi_1{}^i}$, $(i \in I)$ gives an Azumaja R-algebra $A = \mathrm{End}_R(M)$ such that class $[A] \in \mathrm{Br}(K/R)$.

Now we can state the main result.

Theorem (2.3). *Let R be a local normal domain. Then the group* $\mathrm{Br}(K/R)$ *is canonically isomorphic* (*see Proposition* (2.6) *below*) *to the group* $\mathrm{Cl}(\tilde{R})^{\pi_1}/\mathrm{Cl}(R)$.

The theorem follows from two propositions ((2.6), (2.10)) and several lemmas. We need also the following result of Auslander:

Theorem (2.4) [6, Theorem V.12, Corollary VII.3]. *Let T be a normal domain with quotient field Q. Let* $\mathrm{Ref}(T)$ (resp. $\mathrm{Pr}(T)$) *be the monoid of isomorphism classes of reflexive* (resp. *projective*) *T-modules M of finite rank for which* $\mathrm{End}_T(M)$ *is projective. The multiplication is given by* $[M][N] = [(M \otimes_T N)^{**}]$. *Then the sequence*

$$0 \to \mathrm{Cl}(T)/\mathrm{Pic}(T) \xrightarrow{\alpha} \mathrm{Ref}(T)/\mathrm{Pr}(T) \xrightarrow{\beta} \mathrm{Br}(T) \to \mathrm{Br}(Q)$$

is exact, where $\alpha([N]) = [N]$ *and* $\beta([M]) = [\mathrm{End}_T(M)]$.

We begin the proof with

Lemma (2.5). *Let M be a reflexive R-module of finite type. If* $\mathrm{End}_R(M) \approx \mathrm{End}_R(R^n)$, *then $M \approx N \otimes_R R^n$ for some reflexive ideal $N \subset R$.*

Proof. We can represent $1 \in \mathrm{End}_R(R^n)$ as a sum of orthogonal idempotents:

$1 = e_1 + \cdots + e_n$. This gives a decomposition of M into a direct sum of reflexive R-modules of rank 1: $M = e_1M \oplus \cdots \oplus e_nM$. Since $\mathrm{End}_R(e_iM, e_jM)$ is a free R-module of rank 1, $e_iM \approx e_jM$ for all i, j $(1 \leqslant i, j \leqslant n)$. Thus $M \approx N \otimes_R R^n$ for same reflexive ideal $N \subset R$.

PROPOSITION (2.6). *Let S be a Galois covering of R and $M \subset S$ a reflexive ideal such that for every $\sigma \in G = \mathrm{Aut}_R(S)$ the two S-modules σM and M are S-isomorphic. Then $\mathrm{End}_R(M)$ is an Azumaja R-algebra.*

Proof. We shall use the following criterion [10, Lemma 1.2]:

An R-algebra A is an Azumaja algebra iff it is central projective of finite type and $A_\mathfrak{p}$ is an Azumaja $R_\mathfrak{p}$-algebra for every prime ideal $\mathfrak{p} \subset R$ of height 1.

We claim that $A = \mathrm{End}_R(M)$ has center R. Indeed, any element of the center is in $K =$ center $(K \otimes A)$, and any element of A is integral over R by the Cayley–Hamilton theorem. The result now follows from the normality of R.

Now we shall prove that A is a projective R-module. Let

$$P = \bigoplus_{\sigma \in G} (\sigma M) \approx M \otimes_S S^n$$

be an S-module of rank n where $n = |G|$. For any $\sigma \in G$ we can define an additive bijection $\bar{\sigma}$ of P as follows:

$$\bar{\sigma}\colon (0 \cdots 0\ \tau(m)\ 0 \cdots 0) \to (0 \cdots 0\ \sigma\tau(m)\ 0 \cdots 0),$$

where $m \in M$, $\sigma, \tau \in G$, and both elements of P have zero components except at the places τ and $\sigma\tau$ respectively. One easily checks that $\bar{\sigma}(sp) = \sigma(s)\,\bar{\sigma}(p)$ for every $s \in S$, $p \in P$, and $\bar{\tau}\bar{\sigma} = \overline{\tau\sigma}$ for every $\tau, \sigma \in G$. From the descent theory [20, II, Theorem 5.1] follows the existence of an R-module N such that $N \otimes_R S \approx P$. Since $\mathrm{End}_S(M \otimes_S S^n) \approx \mathrm{End}_S(N \otimes_R S)$ and $\mathrm{End}_S(M) \approx S$, $\mathrm{End}_S(N \otimes_R S)$ is a projective S-module. Since $\mathrm{End}_S(N \otimes_R S) \approx \mathrm{End}_R(N) \otimes_R S$, $\mathrm{End}_R(N)$ is a projective R-module. Therefore $\mathrm{End}_R(P)$ is a projective R-module. Hence A is a projective R-module.

Because the last condition of the criterion is obviously satisfied, A is an Azumaja R-algebra.

LEMMA (2.7). *Let $S \supset B$ be a finite etale extension of a normal domain B. If M is a reflexive S-module, then M is reflexive as a B-module. If N is a B-module such that $N \otimes_B S$ is a reflexive S-module, then N is a reflexive B-module. Furthermore, any S-module L is a direct summand of the S-module $S \otimes_R L$.*

Proof. We shall prove only the last assertion. By assumption, there exists an idempotent $e = \sum_\nu x_\nu \otimes y_\nu \in S \otimes_B S$ such that the map: $L \to S \otimes_B L$ given by: $m \mapsto \sum_\nu x_\nu \otimes y_\nu m$ is a splitting monomorphism. Therefore we are done.

(2.8) Suppose M is as in (2.6). From (2.6) and (2.7) it follows that $[\mathrm{End}_R(M)] \in \mathrm{Br}(K/R) \subset \mathrm{Br}(R)$.

LEMMA (2.9). *With the notation as in Proposition* (2.6), *if* $[A] = 0$ *in* $\mathrm{Br}(R)$, *then there exists a reflexive ideal* $L \subset R$ *such that* $M \approx L \otimes_R S$ *is an isomorphism of* S*-modules.*

Proof. By Lemma (2.5), there exists an ideal $L \subset R$ such that $M \approx L \otimes_R R^l$ is an isomorphism of R-modules. Hence the R-module

$$T = \bigcap_{ht\mathfrak{p}} \left(L^* \underset{R}{\otimes} M\right)_{\mathfrak{p}} \qquad (\mathfrak{p} \subset R)$$

is projective, so, by Lemma (2.7), it is projective as an S-module where the action of the ring S comes from the action of S on M. Since $\mathrm{rank}_S(M) = 1$ we have an isomorphism of S-modules $T \approx S$. Since M and L are reflexive R-modules this implies an S-isomorphism: $M \approx L \otimes_R S$.

PROPOSITION (2.10). *Construction of the map*

$$\tau\colon \mathrm{Cl}(\tilde{R})^{\pi_1}/\mathrm{Cl}(R) \to \mathrm{Br}(K/R).$$

Let $[M'] \in \mathrm{Cl}(\tilde{R})^{\pi_1}$ *be a representative of* $\delta \in \mathrm{Cl}(\tilde{R})^{\pi_1}/\mathrm{Cl}(R)$. *Since the ring* $\tilde{R}$ *is Noetherian, there exists a Galois covering* S_i $(i \in I)$ *and a reflexive ideal* $M_i \subset S_i$ *such that* $M' \approx \tilde{R} \otimes_{S_i} M_i$ *and* $\sigma(M_i) \approx M_i$ *as* S_i*-modules for every* $\sigma \in \pi_1$. *By Proposition* (2.6), *the* R*-algebra* $\mathrm{End}_R(M_i)$ *gives an element of the group* $\mathrm{Br}(K/R)$ (*see also* (2.8)). *Define*

$$\tau\colon \delta \mapsto [\mathrm{End}_R(M_i)].$$

The map τ is well defined. Indeed, let S_j $(j \in I)$ is another Galois covering and $M_j \subset S_j$ such that $M' \approx \tilde{R} \otimes_{S_j} M_j$, and $\sigma(M_j) \approx M_j$ as S_j-modules for every $\sigma \in \pi_1$. Then let S_k $(k \in I)$ be a Galois covering such that the pair $(S_k, \mathfrak{m}_k)$ dominates both $(S_i, \mathfrak{m}_i)$ and $(S_j, \mathfrak{m}_j)$. Because $\mathrm{End}_R(S_k \otimes_{S_i} M_i) \approx \mathrm{End}_R[\oplus^t M_i]$, where $t = [S_k\colon S_i]$, we have $[\mathrm{End}_R(S_k \otimes_{S_i} M_i)] = [\mathrm{End}_R(M_i)]$ in $\mathrm{Br}(R)$. This shows that τ is well defined.

LEMMA (2.11). *The map* τ *is surjective.*

Proof. By Auslander's theorem (see Theorem (2.4)) each element of the group $\mathrm{Br}(K/R)$ is of the form $[\mathrm{End}_R(M)]$ for some reflexive R-module M. The ring R is local, hence the algebra $A = \mathrm{End}_R(M)$ contains a maximal commutative subalgebra $S \subset A$, which is a free R-module of $\mathrm{rank}_R(S) = \mathrm{rank}_R(M)$, and $\mathrm{End}_R(M) \otimes_R S \approx \mathrm{End}_S(A)$ [20, Theorem 6.4, p. 104, Proposition 6.1, p. 98]. By Lemma (2.7) M is a reflexive S-module. It has $\mathrm{rank}_S(M) = 1$. Therefore, it remains to show that $\sigma(\tilde{R} \otimes_S M) \approx \tilde{R} \otimes_S M$ as R-modules for every $\sigma \in \pi_1$. For that, it is enough to prove that $\sigma(S_i \otimes_S M) \approx S_i \otimes_S M$ for some $S_i \supset S$ and for all $\sigma \in {\pi_1}^i$ $(i \in I)$.

Fix some $S_i \supset S$ $(i \in I)$. Let T denote the S_i-module $S_i \otimes_S M$. As before, Lemma (2.5) implies that $T \otimes_R S_i \approx \bigoplus^k N$ where $N \subset S_i$ is a reflexive ideal. Through the end of the proof $(\cdots)^*$ will mean $\mathrm{Hom}_{S_i}(\cdots, S_i)$. Tensoring both sides of the last isomorphism with the S_i-module N^* we obtain $T \otimes_R S_i \otimes_S N^* \approx (\bigoplus^k N) \otimes_{S_i} N^*$. Hence $(T \otimes_R N^*)^{**}$ is a free S_i-module. Let $\sum_\nu x_\nu \otimes y_\nu \in S_i \otimes_R S_i$ be a canonical idempotent for the etale extension $S_i \supset R$. We have a split monomorphism of S_i-modules (S_i acts on N^*):

$$\epsilon\colon T \underset{S_i}{\otimes} N^* \to T \underset{R}{\otimes} N^*, \qquad \epsilon(t \otimes n) = \sum_\nu tx_\nu \otimes y_\nu n.$$

Hence $(T \otimes_{S_1} N^*)^{**}$ is a direct summand of a free S_i-module. So it is a free S_i-module of rank 1. Therefore $T \approx N$ as S_i-modules. Since $\sigma(T) \otimes_R S_i \approx \bigoplus^k N$ as S_i-modules for every $\sigma \in {\pi_1}^i$, one can show, as before, that $\sigma(T) \approx N$ as S_i-modules. Thus $\sigma(T) \approx T$ for every $\sigma \in {\pi_1}^i$, and we are done.

LEMMA (2.12). *The map τ is a monomorphism of groups.*

Proof. Let $S = S_i$ $(i \in I)$ be a Galois covering of R, and $[T], [L] \in \mathrm{Cl}(S)^{\pi_1}$. As in the proof of Proposition (2.6), the descent theory implies that $L \otimes_R S \approx \bigoplus^l L \approx N \otimes_R S$ for some R-module N. Hence $\mathrm{End}_R(T \otimes_R L) \otimes_R \mathrm{End}_R(S) \approx \mathrm{End}_R(T \otimes_R N \otimes_R S) \approx \mathrm{End}_R(T \otimes_R N) \otimes_R \mathrm{End}_R(S)$. This implies $\mathrm{End}_R(T \otimes_R L) \otimes_R \mathrm{End}_R(S) \approx \mathrm{End}_R(T \otimes_S S \otimes_R N) \otimes_R \mathrm{End}_R(S) \approx \mathrm{End}_R(T \otimes_S (\bigoplus^l L)) \otimes_R \mathrm{End}_R(S) = \mathrm{End}_R((\bigoplus^l T) \otimes_S L) \otimes_R \mathrm{End}_R(S)$.

This completes the proof of this lemma and Theorem (2.3).

The following proposition was proved by M. Auslander (see remark of MacRae in [24, p. 746]). Because we were unable to find a proof in the literature we shall give a proof here.

PROPOSITION (2.13) (Auslander). *Let B be a normal domain, and M a reflexive B-module of finite type and finite homological dimension such that the B-algebra $A = \mathrm{End}_B(M)$ is projective as B-module. Then M is a projective B-module.*

Proof. By Theorem (2.4) A is an Azumaja B-algebra. We can suppose B to be local. Let S be a maximal commutative subalgebra of the algebra A. It is a splitting ring for A, and the B-module S is free with $rank_B(M) = rank_B(S)$ [20, Theorem 6.4, p. 181]. By Lemma (2.7), M is a reflexive S-module of rank 1 and $M \oplus T \approx S \otimes_R M$ for an S-module T. Hence $hd_S(M) < \infty$. Therefore $[M] = 0$ in $\mathrm{Cl}(S)$ [7, 4.7], so M is a projective S-module. Hence M is a free B-module.

PROPOSITION (2.4). *For any normal domain B the following conditions are equivalent:*

(i) *The domain B is regular of dimension $\leqslant 2$.*

(ii) *All reflexive B-modules of rank 2 are projective.*

(iii) *The ring of endomorphisms of any reflexive B-module of rank $\leqslant 4$ is a projective B-module.*

Proof. (i) $\Rightarrow$ (ii), (iii). This is well known.

(ii) $\Rightarrow$ (i). Suppose B is not regular. Then $\dim(B) \geqslant 2$, and there is a prime ideal $\mathfrak{p} \subset B$ such that the local ring $C = B_\mathfrak{p}$ is not regular. If $\dim(C) = 2$, then it is easy to find two ideals $P = (a, b)$, $Q = (a, b, c)$ in C such that $\operatorname{ann}_C(Q/P) = \mathfrak{p}C$. We have an exact sequence of C-modules

$$0 \to P \to Q \to Q/P \to 0,$$

where, by our assumptions, $hd_C(P)$, $hd_C(Q) \leqslant 1$. Since $hd_C(Q/P) = \infty$, a contradiction. We omit the trival verification that $\dim(C)$ cannot be greater than 2.

(iii) $\Rightarrow$ (ii). Since the ring of endomorphisms of any reflexive B-module of rank 2 is projective, every prime ideal of height 1 in B is invertible. Indeed, if $\mathfrak{q} \subset B$ is a minimal noninvertible prime ideal, then, by assumptions, $\operatorname{End}_B(B \oplus \mathfrak{q})$ is a projective B-module. Because this module contains as a direct summand a nonprojective B-module $\operatorname{Hom}_B(\mathfrak{q}, B)$, we have a contradiction.

Now we can suppose B to be local. Let M be a reflexive B-module of rank 2 such that the B-algebra $A = \operatorname{End}_B(M)$ is projective as B-module. Hence A is an Azumaja B-algebra. The splitting ring $S \subset A$ is semilocal and S is a free B-module of rank 2 [20, Theorem 6.4, p. 181]. Furthermore $[A] \in \operatorname{Br}(S/B)$. We claim that S is a factorial domain. Indeed, any reflexive S-module N of rank 2 is reflexive and has rank 4 as a B-module (Lemma (2.7)). By Lemma (2.7), the S-module $\operatorname{End}_S(N)$ is a direct summand of a projective S-module $\operatorname{End}_S(S \otimes_B N)$. Therefore $\operatorname{End}_S(N)$ is a free S-module. Hence the previous argument shows that S is a factorial domain. Because $\operatorname{End}_S(S \otimes_B M)$ is a trivial Azumaja S-algebra, Lemma (2.5) implies that $S \otimes_B M$ is a free S-module. Therefore M is a free B-module.

I don't know whether the number 4 in (iii) is the best possible.

3. Reflexive Modules over Two-Dimensional Local Normal Domains. Rational Singularities

In this section we study reflexive modules over two-dimensional complete local normal domains. The method we use was introduced by Mumford in his investigation of the Class group of two-dimensional normal singularities [27]. In short, his method consists of the following. We resolve the singularity and look at the Picard group of the neighborhood of the exceptional fiber. Then the Class group is a factor group of this Picard group. Using the methods of Grothendieck's formal geometry Danilov ([11], etc.) proved some deep results on the Class group of a complete domain. For a survey of his results see [15], [23]. This method was especially useful when the singularity is rational (see Artin [1, 2], Brieskorn [9], Lipman [21]). Although their results give the description of the Class group of complete rings of some rational singularities, little is known about reflexive modules of rank $n > 1$.

The case of reflexive modules of rank 1 is much simpler. The Class group has a natural group structure which allows to introduce a geometric structure. This doesn't work for the set of reflexive modules of rank $n > 1$.

Recently Boutot [8] applied Artin's criterion of representability [4] to the investigation of the Class group. This criterion and the deformation theory (Schlessinger's local criterion) suggest a new approach to the investigation of reflexive modules over complete local rings.

The main results of this section are Proposition (3.5) and Theorem (3.9). As a corollary we get a desciption of reflexive modules over complete two-dimensional cone (see (3.11), (3.13), and (3.14)).

(3.1) Let X be an integral scheme. A coherent $\mathcal{O}_X$-module is called reflexive if $M \approx M^{**}$. We denote by $\mathscr{R}(X)$ (resp. $\mathscr{R}_n(X)$) the set of all reflexive sheaves (resp. all reflexive sheaves of rank n) on X. Let $\operatorname{Vect}(X)$ (resp. $\operatorname{Vect}_n(X)$) denote the set of all vector bundless (resp. all vector bundles of rank n) on X. If $X = \operatorname{Spec}(A)$ we also use the notations $\mathscr{R}(A)$, $\operatorname{Vect}(A)$,..., instead of $\mathscr{R}(X)$, $\operatorname{Vect}(X)$,...,

(3.2) Let $(A, \mathfrak{m})$ be a complete local normal domain of dimension 2, $V = \operatorname{Spec}(A)$.

The morphism $f\colon X \to V$ is called a resolution of the singularity of V if

(a) X is a regular scheme,

(b) f is a proper morphism,

(c) f induces an isomorphism: $f^{-1}(V\backslash\{\mathfrak{m}\}) \simeq V\backslash\{\mathfrak{m}\}$.

(3.3) Let $f\colon X \to V$ be a resolution of the singularity. We define a natural map of sets

$$\bar{f}\colon \operatorname{Vect}(X) \to \mathscr{R}(A)$$

as follows. Let $M \in \operatorname{Vect}(X)$. The restriction of the sheaf M to $f^{-1}(V\backslash\{\mathfrak{m}\})$ gives a locally free sheaf on V_1 (points of codim $\leqslant 1$ in V), hence a reflexive A-module.

The map $\bar{f}$ is surjective. Indeed, any reflexive A-module of finite type gives a unique vector bundle on $f^{-1}(V\backslash\{\mathfrak{m}\})$. Since X is two-dimensional and regular, we can extend this bundle (nonuniquely) to a vector bundle on X.

Thus, we can divide the problem about reflexive modules over the domain A into two parts: investigation of Vect(X) and invesigation of of the "kernel" of the map $\bar{f}$. In this section we will deal only with the first part of the problem.

We collect together some known facts.

(3.4) It is known (Abhyankar, Hironaka; see [22]) that the singularity of the normal domain A can be resolved by a monoidal transformation, namely, there exists an ideal $I \subset A$ with $V(I) = \{\mathfrak{m}\}$ such that the monoidal transformation with center in I:

$$f\colon X \to V = \operatorname{Spec}(A)$$

is a resolution of the singularity. Thus $I\mathcal{O}_X$ is an invertible very ample [18, No. 8, II, (4.4.2)] sheaf on the V-scheme $X = \operatorname{Proj}(\bigoplus_{n=0}^{\infty} I^m)$. The fiber of f

$$Y = X \times_V \operatorname{Spec}(A/I) = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty}\right) I^m/I^{m+1}$$

is a 1-dimensional scheme. The natural morphism $\varphi\colon Y \to \operatorname{Spec}(A/I)$ is projective and $\mathcal{O}_Y = \mathcal{O}_X/I\mathcal{O}_X \mid Y$.

Let $N_Y = i^*(I\mathcal{O}_X/I^2\mathcal{O}_X)$ be a conormal sheaf on Y where $i\colon Y \to X$. The sheaf N_Y is a very ample sheaf relative to the map φ. Thus by Serre's theorem [18, No. 11, III, (2.2.1)], there is an integer $k_0 > 0$ such that for any $k \geqslant k_0$,

$$H^1(Y, N_Y{}^k) = 0.$$

We adopt the following notations:

$$Y_l = X \times_V \operatorname{Spec}(A/I^l), \qquad \mathcal{O}_l = \mathcal{O}_{Y_l} = \mathcal{O}_X/I^l\mathcal{O}_X \mid Y_l \qquad \text{for } l > 0;$$

$\hat{X}$ the completion of X along $I\mathcal{O}_X$ [18, No. 4, I, (10.10.1)]; and F the underlying space of all the Y_l.

Proposition 3.5. *Let A be a complete local normal two-dimensional domain. Then with the above notations, the map*

$$H^1(X, \mathcal{G}\ell(n, \mathcal{O}_X)) \to H^1(F, \mathcal{G}\ell(n, \mathcal{O}_l))$$

is bijective for all $l \geqslant k_0$, $n \geqslant 1$.

Proof. First we prove a known fact that the map

$$H^1(\hat{X}, \mathcal{G}\ell(n, \mathcal{O}_{\hat{X}})) \to \varprojlim_l H^1(Y_l, \mathcal{G}\ell(n, \mathcal{O}_l))$$

is bijective (cf. [3, Proof of Theorem 3.5]). This follows from Grothendieck's existence theorem [18, No. 11, III, (5.1)]. Indeed, a compatible system of elements $a_l \in H^1(Y_l, \mathscr{G}(n, \mathcal{O}_l))$ is determined by locally free sheaves L_l and isomorphisms $L_l \otimes \mathcal{O}_{l-1} \approx L_{l-1}$, i.e., by a formal sheaf. This formal sheaf is induced by a locally free sheaf $\hat{L}$ on $\hat{X}$, by Grothendieck's theorem, and $\hat{L}$ is necessarily locally free. Thus this map is surjective. Now suppose $\hat{L}$ is a locally free sheaf on X such that L_l is free for each l. The modules $H^0(Y_l, L_l)$ are of finite length for each l. Hence the images of the maps

$$H^0(Y_m, L_m) \to H^0(Y_l, L_l)$$

$(m \geqslant l)$ are constant for large m, say equal to $M_l \subset H^0(Y_l, L_l)$. The M_l form an inverse system of modules whose maps $M_l \to M_{l-1}$ are surjective, and clearly [18, No. 11, III, (4.1.5)]

$$\varprojlim M_l = \varprojlim H^0(Y_l, L_l) \approx H^0(\hat{X}, \hat{L})$$

Since L_m is free, it contains sections $s_m^1, \ldots, s_m^n$ whose determinant is nowhere zero. This being true for all m, the module M_0 must contain sections $s_0^1, \ldots, s_0^n$ of L_0 with nowhere zero determinant. These sections lift successively to M_l for each l, hence to sections $\hat{s}^1, \ldots, \hat{s}^n \in H^0(\hat{X}, \hat{L})$. Since the determinant of these sections $\hat{s}^i$ is not zero on Y_0, it is nowhere zero. This concludes the proof of the first assertion.

Secondly, Grothendieck's existence theorem implies that the functor $\mathscr{F} \to \hat{\mathscr{F}}$ is an equivalence between the category of coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_{\hat{X}}$-modules [18, No. 11, III, (5.1.6)]. Thus, injectivity of the map

$$H^1(X, \mathscr{G}\ell(n, \mathcal{O}_X)) \to H^1(\hat{X}, \mathscr{G}\ell(n, \mathcal{O}_X))$$

would imply its bijectivity. Let us prove that this map is injective. If L is a locally free sheaf on X such that the induced sheaf $\hat{L}$ is free, then there are sections $\hat{s}^1, \ldots, \hat{s}^n \in H^0(\hat{X}, \hat{L})$ which have nowhere zero determinant. By [18, No. 11, III, (5.1.2)], $H^0(\hat{X}, \hat{L}) = H^0(X, L)$. Let $s^1, \ldots, s^n \in H^0(X, L)$ be the images of $\hat{s}^1, \ldots, \hat{s}^n$. Then the determinant of s^i is nowhere zero. Thus L is free.

Now let $\mathscr{M}(n, N_Y^l)$ $(l > 0)$ be a sheaf of Abelian groups on F whose stalk at a point $x \in F$ is an Abelian group of $n \times n$ matrices over $(N_Y^l)_x$. We have an exact sequence of sheaves of groups on F

$$0 \to \mathscr{M}(n, N^l) \xrightarrow{\mu} \mathscr{G}\ell(n, \mathcal{O}_{l+1}) \to \mathscr{G}\ell(n, \mathcal{O}_l) \to 1,$$

where μ is given by $s \to 1 + s$ for any section s of the sheaf $\mathscr{M}(n, N)_Y^l$. This sequence yields a cohomology sequence

$$H^1(F, \mathscr{M}(n, N_Y^l)) \to H^1(F, \mathscr{G}\ell(n, \mathcal{O}_{l+1})) \to H^1(F, \mathscr{G}\ell(n, \mathcal{O}_l)).$$

If $l \geqslant k_0$, then $H^1(F, \mathscr{M}(n, N_Y^l)) = 0$ by (3.4). Therefore, the proposition will follow from

LEMMA (3.6). *Let Z be a one-dimensional Noetherian scheme. Let $\mathcal{O}_{Z_0} = \mathcal{O}_Z/I$ be a structure sheaf of a closed subscheme $Z_0 \subset Z$ defined by a sheaf of ideals $I \subset \mathcal{O}_Z$ with $I^2 = 0$. Then for any locally free $\mathcal{O}_{Z_0}$-module M_0 there is a locally free $\mathcal{O}_Z$-module M such that $M \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_0} = M_0$.*

Proof. By [19, IV, Proposition 3.15], an obstruction to the lifting of the locally free $\mathcal{O}_{X_0}$-module M_0 lies in the group $\mathrm{Ext}^2_{\mathcal{O}_{Z_0}}(M_0, ...)$. We claim that this group is trivial. In fact, since $H^2(Z_0, ...) = 0$, the spectral sequence of composition functors [16]:

$$H^p(Z_0, \mathcal{E}xt^q_{\mathcal{O}_{Z_0}}(M_0, ...)) \Rightarrow \mathrm{Ext}^n_{\mathcal{O}_{Z_0}}(M_0, ...)$$

implies $\mathrm{Ext}^2_{\mathcal{O}_{Z_0}}(M_0, ...) = 0$.

DEFINITION (3.7) (cf. [2], [21]). A normal local domain of dimension 2 is said to have a rational singularity if there exists a desingularization $X \to \mathrm{Spec}(A)$ such that $H^1(X, \mathcal{O}_X) = 0$.

(3.8) Suppose now that $(A, \mathfrak{m})$ is a complete local normal domain which has a rational singularity and algebraically closed residue field k.

Let $f: X \to V = \mathrm{Spec}(A)$ be the minimal desingularization [21, Theorem 4.1]. The reduced fiber $F = f^{-1}(\{\mathfrak{m}\})_{\mathrm{red}}$ consists of k-smooth ration curves $\{X_i\}_{1 \leqslant i \leqslant p}$ [2, Proposition 1]. Let Z denote the fundamental cycle of F [2, p. 132]. Obviously $\mathrm{Supp}(Z) = F$.

Artin proved that $\mathcal{O}(-nZ) \approx \mathfrak{m}^n\mathcal{O}_X$ and $H^1(Z, J_n) = 0$, where $J_n = \mathrm{Ker}(\mathcal{O}_{(n+1)Z} \to \mathcal{O}_{nZ}) \approx \mathcal{O}(-nZ) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ $(n \geqslant 1)$ [2].

THEOREM (3.9). *With the notation of* (3.8), *the map*

$$H^1(X, \mathcal{G}\ell(n, \mathcal{O}_X)) \to H^1(F, \mathcal{G}\ell(n, \mathcal{O}_F))$$

is bijective for any integer $n > 0$.

Proof. One can show exactly as in Proposition (3.5) that the map

$$H^1(X, \mathcal{G}\ell(n, \mathcal{O}_X)) \to H^1(F, \mathcal{G}\ell(n, \mathcal{O}_Z))$$

is bijective. Instead of the ideal I we should consider the ideal $\mathfrak{m} \subset A$. In general, the morphism f is no longer a monoidal transformation with center in $\mathfrak{m}$ but that doesn't affect the proof.

Thus, it remains to show that the natural map

$$H^1(F, \mathcal{G}\ell(n, \mathcal{O}_Z)) \to H^1(F, \mathcal{G}\ell(n, \mathcal{O}_F))$$

is bijective.

Let $Z \geqslant Z^1 > Z^2 \geqslant F$ be any two positive cycles with support in F such that in the exact sequence of sheaves,

$$0 \to K \to \mathcal{O}_{Z^1} \to \mathcal{O}_{Z^2} \to 0,$$

the kernel K satisfies condition $K^2 = 0$. We have the diagram of sheaves of groups on F

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(n, K) \arrow[r] \arrow[d, "\mu"] & \mathcal{M}(n, \mathcal{O}_{Z^1}) \arrow[r] & \mathcal{M}(n, \mathcal{O}_{Z^2}) \arrow[r] & 0 \\
1 \arrow[r] & \mathcal{G}\ell(n, K) \arrow[r] & \mathcal{G}\ell(n, \mathcal{O}_{Z^1}) \arrow[r] & \mathcal{G}\ell(n, \mathcal{O}_{Z^2}) \arrow[r] & 0,
\end{tikzcd}$$

where isomorphism μ is given by $t \to 1 + t$ for any section t of the sheaf of matrices $\mathcal{M}(n, K)$. This diagram yields the diagram

$$H^0(F, \mathcal{M}(\mathcal{O}_{Z^1})) \longrightarrow H^0(F, \mathcal{M}(\mathcal{O}_{Z^2}))$$
$$H^0(F, \mathcal{G}\ell(\mathcal{O}_{Z^1})) \longrightarrow H^0(F, \mathcal{G}\ell(\mathcal{O}_{Z^2}))$$

$$\begin{tikzcd}
{} \arrow[r, "\alpha"] & H^1(F, \mathcal{M}(K)) \arrow[r] \arrow[d, "\sigma"] & H^1(F, \mathcal{M}(\mathcal{O}_{Z^1})) \arrow[r] & H^1(F, \mathcal{M}(\mathcal{O}_{Z^2})), \\
{} \arrow[r, "\beta"] & H^1(F, \mathcal{G}\ell(K)) \arrow[r, "\gamma"] & H^1(F, \mathcal{G}\ell(\mathcal{O}_{Z^1})) \arrow[r, "\tau"] & H^1(F, \mathcal{G}\ell(\mathcal{O}_{Z^2}))
\end{tikzcd}$$

By Lemma (3.6) τ is surjective. So, it is enough to show that β is surjective.

Since $H^1(F, \mathcal{O}_{Z^1}) = H^1(F, \mathcal{O}_{Z^2}) = 0$ [1, Theorem (1.7); 2, Proposition 1], the map α is onto. Every element of $H^1(F, \mathcal{G}\ell(n, K))$ is a sum of elements of the form $\sigma(\omega_{u,v})$, where $\omega_{u,v} \in \mathbf{M}(n, H^1(F, K)) = H^1(F, \mathcal{M}(n, K))$ is a matrix with entries zero except at the place (u, v). Let $\omega = \sigma(\omega_{u,v})$. We have two cases.

Case $u \neq v$. Let s be a section of $\mathcal{M}(n, \mathcal{O}_{Z^2})$ such that $\alpha(s) = \omega_{u,v}$. Choose an open covering $\{U_j\}$ of F such that s can be lifted to sections $s_j \in \Gamma(U_j, \mathcal{M}(n, \mathcal{O}_{Z^1}))$. Then $s_j - s_k \in \Gamma(U_j \cap U_k, \mathcal{M}(n, K))$ and $\alpha(s)$ is represented by 1-cocycle $\{(s_j - s_k)\}$. We can suppose that all s_j are matrices with zero entries except at the place (u, v). Further, $1 + s \in H^0(F, \mathcal{G}\ell(n, \mathcal{O}_{Z^2}))$. We claim that $\beta(1 + s) = \omega$. Indeed, the section $1 + s$ can be lifted to the sections $1 + s_j$ of $\mathcal{G}\ell(n, \mathcal{O}_{Z^1}) \mid U_j$, hence $\beta(1 + s)$ is represented by the cocycle $\{(1 + s_j)/(1 + s_k)\}$. Because $u \neq v$, $(1 + s_j - s_k)(1 + s_k) = 1 + s_j$, so $1 + s_j - s_k = (1 + s_j)/(1 + s_k)$. Thus $\beta(1 + s) = \omega$.

Case $u = v$. The element $\gamma(\omega)$ corresponds to a locally free $\mathcal{O}_{Z^1}$-module decomposable into the direct sum of $\mathcal{O}_{Z^1}$-modules of rank 1. It is known that $H^1(F, \mathcal{O}_{Z^1}^*) \approx H^1(F, \mathcal{O}_{Z^2}^*) \approx \mathbf{Z}^p$ [1, Theorem 1.7]. Therefore, if $\gamma(\omega) \neq 1$, then $\tau\gamma(\omega) \neq 1$, a contradiction. Hence $\gamma(\omega) = 1$, and $\omega \in \mathrm{Im}(\beta)$.

Remark (3.10). Let D be a field or an excellent Dedekind domain. Let $(A, \mathfrak{m})$ be an excellent local normal two-dimensional domain which is a Henselization of an D-algebra of finite type at a prime ideal. Let $X \to \operatorname{Spec}(A)$ be a resolution of singularity by blowing up an ideal $I \subset A$ as in (3.4).

Denote by $\hat{A}$ the I-adic completion of A. Let

$$X_l = X \underset{\operatorname{Spec}(A)}{\times} \operatorname{Spec}(A/I^l), \qquad l \geqslant 1.$$

Artin's Approximation Theorem [3, Theorems (1.12) and (3.5)] and Proposition (3.6) imply that the natural map $\mathscr{R}(X) \to \mathscr{R}(X_l)$ is bijective for $l \gg 0$, and the natural map $\mathscr{R}(X) \to \mathscr{R}(X_1)$ is surjective (see Lemma 3.6)).

Moreover, the natural diagram

$$\begin{array}{ccc} \mathscr{R}(X) & \longrightarrow & \mathscr{R}(A) \\ \downarrow & & \downarrow \varphi \\ \mathscr{R}(X_l) & \longrightarrow & \mathscr{R}(\hat{A}) \end{array} \qquad l \gg 0,$$

is commutative, hence the map φ is surjective. Of course, the last result follows also from Elkik's Approximation Theorem [14, Theorem 3].

Theorem (3.9) also admits a modification to the Henselian case.

COROLLARY (3.11). *Let $A = k[[X, Y, Z]]/(X^2 + Y^2 + Z^2)$ be a cone where k is an algebraically closed field of* $\operatorname{char}(k) \neq 2$.

Then every reflexive A-module is a direct sum of reflexive A-modules of rank 1. *Such decomposition is unique.*

Proof. We can resolve singularity of $\operatorname{Spec}(A)$ by blowing up the closed point. The exceptional fiber of this resolution consists only of one rational smooth k-curve. The result now follows from (3.3), (3.9), and the Krull–Schmidt theorem.

Remark (3.12). There exists an indecomposable reflexive module of rank >1 over any two-dimensional nonregular factorial domain, in particular, over rational singularities of type E_8 [21, Theorem 25.1)]. On the other hand, we conjecture the following:

Let $(A, \mathfrak{m})$ be a complete local normal domain of dimension 2 and $k = A/\mathfrak{m}$ be algebraically closed of $\operatorname{char}(k) \neq 2$. Then A is a cone as in (3.11) iff every reflexive A-module is decomposable into a direct sum of rank 1 submodules.

COROLLARY (3.13). *Let A be as in* (3.11). *Then the module of k-derivation of A has the form*

$$D_k^*(A) \approx \mathfrak{p} \oplus \mathfrak{p},$$

where $\mathfrak{p} \subset A$ is a nonprincipal prime ideal of height 1.

Proof. It is well known that the ring A is normal and $\mathrm{Cl}(A) = \mathbf{Z}/2\mathbf{Z}$ [15]. By Corollary (3.11), $D_k^*(A) \approx J_1 \oplus J_2$ where $J_1, J_2 \subset A$ are reflexive ideals. By [1, Theorem 2.7], the module $\Lambda^2 D_k(A)$ of two-dimensional differential forms is free. So $(\Lambda^2 D_k(A))^* \approx (\Lambda^2 D_k^*(A))^{**} \approx A$. Hence $J_1 \approx J_2$. The Scheja–Storch theorem on the module of derivations of hypersurfaces [13, III, b] (or a direct computation) implies that $D_k^*(A)$ cannot be a free A-module. Hence we are done.

PROPOSITION (3.14). *Let $A = \mathbb{R}[[X, Y, Z]]/(X^2 + Y^2 + Z^2)$ be a complet real cone with quotient field Q. Then we have an A-isomorphism*

$$D_{\mathbb{R}}^*(A) \approx \mathfrak{p},$$

where $\mathfrak{p}$ is a non-principal prime ideal of height 1 in $A_{\mathbb{C}}$, the complexification of A.

Proof. By Corollary (3.13), $D_{\mathbb{R}}^*(A) \otimes_A A_{\mathbb{C}} \approx \mathfrak{p} \oplus \mathfrak{p}$ as A-modules. Hence $C = \mathrm{End}_A(D_{\mathbb{R}}^*(A))$ is a free A-module, so a nontrivial Azymaja A-algebra, and $[C] \in \mathrm{Br}(Q/A)$ (cf. (2.4)). It is known that $\mathrm{Br}(Q/A) = \mathbf{Z}/2\mathbf{Z}$ (see also Theorem (2.3)). Furthermore, by Proposition (2.6), $[\mathrm{End}_A(\mathfrak{p})]$ is a nontrivial element in $\mathrm{Br}(Q/A)$. Since $\mathfrak{p}$ and $D_{\mathbb{R}}^*(A)$ have the same rank over A, Auslander's theorem (see (2.4)) implies that they are isomorphic as A-modules.

4. The Brauer Group of a Normal Singularity

(4.1) Let $(R, \mathfrak{m})$ be a local normal domain with quotient field K and algebraically closed residue field k. Let $X = \mathrm{Spec}(R)$. Throughout this section we consider the etale topology on X. We denote by R^h the Henselization of the ring R, by $\mathbb{G}_m$ the sheaf of units. If D is an Abelian group and l a prime number, we denote by $D(l)$ (resp. $t(D)$) the l-torsion (resp. torsion) part of D.

(4.2) From [17, II, 1] and Theorem (2.3) follows the existence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Cl}(R^h)/\mathrm{Cl}(R) \arrow[r] & H^2_{et}(X, \mathbb{G}_{m,X}) \arrow[r, "j_2"] & \mathrm{Br}(K) \\
0 \arrow[r] & \mathrm{Cl}(\tilde{R})^{\pi_1}/\mathrm{Cl}(R) \arrow[r, "\tau"] \arrow[u, "\psi"] & \mathrm{Br}(R) \arrow[r] \arrow[u, "\varphi"] & \mathrm{Br}(K), \arrow[u, equal]
\end{tikzcd}$$

where τ is the map defined in Theorem (2.3), φ is the natural embedding (see [17, I, 2] or [20, V, 2]) and ψ is the monomorphism which makes the diagram commutative.

Remark (4.3). It is known that $\mathrm{Cl}(R^h) = \mathrm{Cl}(\hat{R})$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R (see [14, Theorem 3]). We don't know whether $\mathrm{Cl}(\hat{R}) =$

$\mathrm{Cl}(R^h)$ in general. There are many examples where $t(\mathrm{Cl}(\hat{R})/\mathrm{Cl}(R)) \neq 0$. Consider any two-dimensional local factorial domain A with nonrational singularity and an algebraically closed residue field of characteristic 0.

Remark (4.4). Let B be a commutative Noetherian ring of characteristic $p > 0$ with quotient ring F, $V = \mathrm{Spec}(B)$. Then, by [31], $H^2_{et}(V, \mathbb{G}_{m,V})(p) = \mathrm{Br}(B)(p)$ and there are many examples where $\mathrm{Br}(F/B) \neq 0$.

(4.5) Now we recall a result of Rim [29]. Let $S \supset T$ be an unramified integral extension of a local normal domain whose quotient field extension is Galois with the finite Galois group G. Rim obtained an exact sequence of groups

$$0 \to \mathrm{Cl}(S)^G/\mathrm{Cl}(T) \xrightarrow{\tau'} \mathrm{Br}(S/T) \to \bigcap_{ht\mathfrak{p}=1} \mathrm{Br}(S_\mathfrak{p}/T_\mathfrak{p})$$
$$\to H^1(G, \mathrm{Cl}(S)) \to H^3(G, S^*).$$

His proof is purely cohomological and doesn't give an explicit form of the map τ'.

(4.6) From now on, in addition to the hypothesis of (4.1), R will be two-dimensional such that there exists a desingularization $f\colon Y \to X$.

We have the Leray spectral sequence

$$H^p_{et}(X, R^q f_*(\mathbb{G}_{m,Y})) \Rightarrow H^n_{et}(Y, \mathbb{G}_{m,Y}),$$

and the exact sequence of terms of low degree

$$H^1_{et}(Y, \mathbb{G}_{m,Y}) \to H^0_{et}(X, R^1 f_*(\mathbb{G}_{m,Y})) \to H^2_{et}(X, f_*(\mathbb{G}_{m,Y})) \xrightarrow{j_2} H^2_{et}(Y, \mathbb{G}_{m,Y})$$

The normality of R implies that $f_*(\mathbb{G}_{m,Y}) = \mathbb{G}_{m,X}$. Since Y is regular and $\dim Y = 2$, $H^2_{et}(Y, \mathbb{G}_{m,Y}) = \mathrm{Br}(Y)$ [17, II, Corollary 2.2], and the natural map $\mathrm{Br}(Y) \to \mathrm{Br}(K)$ is a monomorphism. Hence $\mathrm{Ker}(j_2) = \mathrm{Ker}(j_1)$ (see (4.3)). It is well known that $H^1_{et}(X, \mathbb{G}_{m,X}) = \mathrm{Pic}(X) = 0$ and $H_{et}(Y, \mathbb{G}_{m,Y}) = \mathrm{Pic}(Y)$ [5, exp. IX, Theorem 3.3]. By well-known result of Artin $R^i f_*(\mathbb{G}_{m,Y}) = 0$ for $i \geq 2$. Hence j_2 is surjective.

It would be nice to know for which local normal domains A the natural map

$$\mathrm{Br}(A) \to \bigcap_{ht\mathfrak{p}=1} \mathrm{Br}(A_\mathfrak{p})$$

is surjective (see [17, Corollary 6.2]).

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